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POLYHEDRAL HARMONICS

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1. Polytopes and the Mean Value Property

Let P be any (not necessarily convex nor connected) solid polytope in the n-dimensional Euclidean space $\mathbb{R}^n$. Here a solid polytope means a finite union of closed convex polytopes, and a closed convex polytope means a finite intersection of closed half-spaces in $\mathbb{R}^n$ which is bounded and contains an interior point. For $k=0,1,\ldots,n$, let $P(k)$ be the k-skeleton of P, and μ_k the k-dimensional Euclidean measure on $P(k)$, where μ_0 is the Dirac measure on the vertices of P. We denote by $|P(k)|=\mu_k(P(k))$ the total measure of $P(k)$.

Definition 1.1. Let Ω be an open set in $\mathbb{R}^n$. A $\mathbb{C}$-valued continuous function f in Ω is said to satisfy the $P(k)$-*mean value property* if for each $x\in\Omega$ there is a sufficiently small positive constant $r_x>0$ such that

$$\text{(MVP)} \qquad f(x)=\frac{1}{|P(k)|}\int_{P(k)} f(x+ry)\,d\mu_k(y)$$

holds for any $0<r<r_x$, where r_x depends on $x\in\Omega$ in such a manner that $\inf_{x\in K} r_x>0$ for any compact subset K of Ω. Let $\mathcal{H}_{P(k)}(\Omega)$ denote the set of all such functions. Any $f\in\mathcal{H}_{P(k)}(\Omega)$ is refered to as a $P(k)$-*harmonic function* in Ω.

It is easy to see that $\mathcal{H}_{P(k)}(\Omega)$ forms a linear space containing the constant functions. Characterizing the function space $\mathcal{H}_{P(k)}(\Omega)$ is an interesting problem which has a long history and has attracted many authors' attention. Here we only refer to the papers [1][2][4][5][6][7][8][15][16]. See the references in [10] for more extensive literature. Neverthless, our knowledge about the space is still very poor. In fact, the problem has been solved satisfactorily for only a few specific polytopes, and what can be said in general remains quite restricted.

In 1962, A. Friedman and W. Littman [8] proposed the following problem.

Problem 1.2. Is $\mathcal{H}_{P(k)}(\Omega)$ finite dimensional ?

This problem had been open until recently when the author was able to solve it affirmatively (see [10]). Originally Friedman and Littman [8] assumed the convexity of P and $k=0,n-1,n$, but these assumptions were unnecessary. The author's recent results are summarized in the following theorem.

Theorem 1.3. *Let P be any (not necessarily convex nor connected) solid polytope in $\mathbb{R}^n$, and let Ω be any open subset of $\mathbb{R}^n$. For any $k = 0, 1, \dots, n$,*

(1) *the restriction map $\mathcal{H}_{P(k)}(\mathbb{R}^n) \to \mathcal{H}_{P(k)}(\Omega)$ is an isomorphism, and hence $\mathcal{H}_{P(k)}(\Omega)$ is independent of the domain Ω,*
(2) *$\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of polynomials,*
(3) *a basis of $\mathcal{H}_{P(k)}(\Omega)$ can be taken from homogeneous polynomials,*
(4) *$\mathcal{H}_{P(k)}(\Omega)$ admits a structure of $\mathbb{C}[\partial]$-module, where $\mathbb{C}[\partial]$ is the ring of linear partial differential operators with constant coefficients, and*
(5) *if the complete symmetry group $G \subset O(n)$ of P is irreducible, then $\mathcal{H}_{P(k)}(\Omega)$ is a finite-dimensional linear space of harmonic polynomials.*

Let $\mathcal{H}(\Omega)$ be the set of all (usual) harmonic functions in Ω. Then the above theorem offers a sharp contrast between $\mathcal{H}_{P(k)}(\Omega)$ and $\mathcal{H}(\Omega)$. Indeed, $\mathcal{H}_{P(k)}(\Omega)$ is independent of Ω, while $\mathcal{H}(\Omega)$ depends heavily on Ω, the dependence comming partly from the presence of natural boundaries; $\mathcal{H}_{P(k)}(\Omega)$ is finite dimensional, while $\mathcal{H}(\Omega)$ is infinite dimentional; $\mathcal{H}_{P(k)}(\Omega)$ contains only polynomials, while $\mathcal{H}(\Omega)$ contains more transcendental functions. By a theorem of Gauss, the usual harmonic functions are characterized by the mean value property with respect to a ball (or a sphere). So the theorem implies that a polytope and a ball are completely different as far as the mean value property is concerned.

Since $\mathcal{H}_{P(k)}(\Omega)$ is independent of Ω, we can use the simplified notation $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$. The third assertion of Theorem 1.3 yields the direct sum decomposition:

$$\mathcal{H}_{P(k)} = \bigoplus_{m \geq 0}^{\text{finite}} \mathcal{H}_{P(k)}(m),$$

where $\mathcal{H}_{P(k)}(m)$ is the linear space of all homogeneous polynomials of degree m satisfying the $P(k)$-mean value property (MVP). In view of Theorem 1.3, the following problem seems interesting.

Problem 1.4.

(1) Determine $\dim \mathcal{H}_{P(k)}$ and constract a basis of $\mathcal{H}_{P(k)}$.
(2) Determine $\dim \mathcal{H}_{P(k)}(m)$ and construct a basis of $\mathcal{H}_{P(k)}(m)$.
(3) Determine the structure of $\mathcal{H}_{P(k)}$ as a $\mathbb{C}[\partial]$-module.

We give an example to demonstrate what is relevant in this problem.

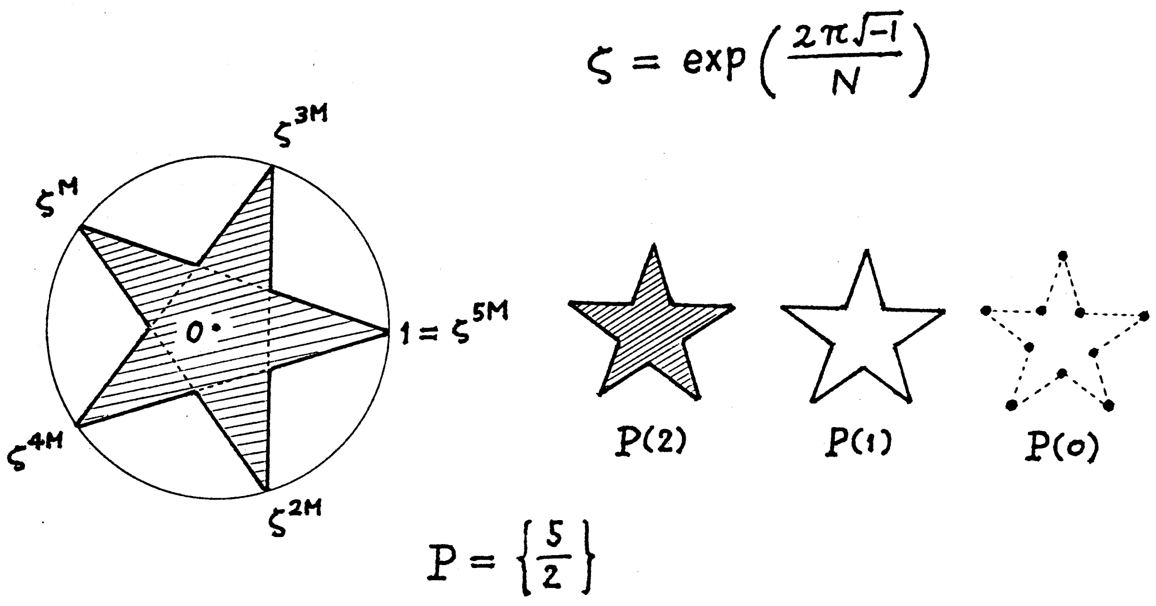
Example 1.5. Let $P = \{N/M\}$ be the regular star-polygon in $\mathbb{R}^2$ with center at the origin, where M and N are coprime natural numbers. See Coxeter's book [3] for its definition. The case $P = \{5/2\}$ is demonstrated in the figure below together with its skeltetons. We remark that if $M = 1$ then $P = \{N\}$ is a regular convex N-gon. The dimension of $\mathcal{H}_{P(k)}$ is given by

$$\dim \mathcal{H}_{P(k)} = 2N \qquad (k = 0, 1, 2).$$

Let (x, y) be an orthonormal coordinate system of $\mathbb{R}^2$ such that P is symmetric with respect to the x-axis. We set $z = x + \sqrt{-1}y$. Then,

$$\mathcal{H}_{P(k)}(m) = \begin{cases} \mathbb{C} & (m = 0), \\ \mathbb{C}z^m \oplus \mathbb{C}\bar{z}^m & (1 \le m \le N-1), \\ \mathbb{C}\mathrm{Im}(z^N) & (m = N), \\ \{0\} & (m \ge N+1), \end{cases}$$

where $\mathrm{Im}(z^N)$ is the imaginary part of z^N. As a $\mathbb{C}[\partial]$-module, $\mathcal{H}_{P(k)}$ is generated by the single element $\mathrm{Im}(z^N)$.



2. Partial Differential Equations

The classical mean value property (with respect to a ball or a sphere) is characterized by the Lapace equation $\Delta f = 0$. The $P(k)$-mean value property can also be characterized in terms of partial differential equations, though, not by a single equation but by a *system* of infinitely many equations.

In order to describe this system, we introduce some notations. For $j = 0, 1, \dots, n$, let $\{P_{i_j}\}_{i_j \in I_j}$ be the set of j-dimensional faces of P, H_{i_j} the j-dimensional affine subspace of $\mathbb{R}^n$ containing P_{i_j}, $\pi_{i_j} : \mathbb{R}^n \to H_{i_j}$ the orthogonal projection from $\mathbb{R}^n$ down to the subspace H_{i_j}. Let $p_{i_j} \in \mathbb{R}^n$ be the vector (or point) in $\mathbb{R}^n$ defined by

$$p_{i_j} = \pi_{i_j}(0) \in H_{i_j}.$$

We remark that $P_{i_0} = H_{i_0} = \{p_{i_0}\}$ for any $i_0 \in I_0$ and that $H_{i_n} = \mathbb{R}^n$ and $p_{i_n} = 0$ for any $i_n \in I_n$. For $i_j \in I_j$ and $i_{j+1} \in I_{j+1}$ we write $i_j \prec i_{j+1}$ if P_{i_j} is a face of $P_{i_{j+1}}$. For $i_j \prec i_{j+1}$ let $\mathbf{n}_{i_j i_{j+1}}$ be the outer unit normal vector of $\partial P_{i_{j+1}}$ in $H_{i_{j+1}}$ at the face P_{i_j}. It is easy to see that the vector $p_{i_j} - p_{i_{j+1}}$ is parallel to $\mathbf{n}_{i_j i_{j+1}}$, so that one can define the *incidence number* $[i_j : i_{j+1}] \in \mathbb{R}$ by the relation:

$$p_{i_j} - p_{i_{j+1}} = [i_j : i_{j+1}]\mathbf{n}_{i_j i_{j+1}}.$$

Let $I(k)$ be the index set defined by

$$I(k) = \{i = (i_0, i_1, \ldots, i_k)\, ;\ i_j \in I_j, i_0 \prec i_1 \prec \cdots \prec i_k\}.$$

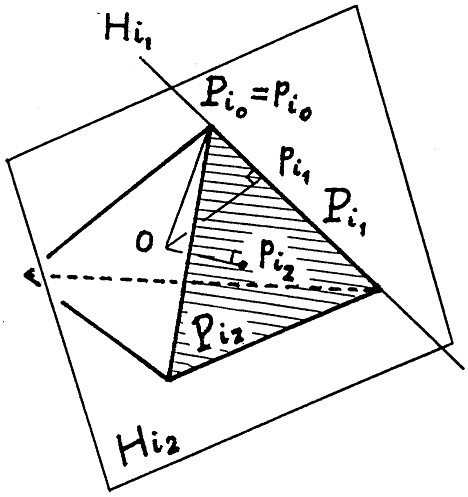
Each element $i \in I(k)$ is refered to as a *k-flag*. For any k-flag $i = (i_0, i_1, \ldots, i_k) \in I(k)$, we set

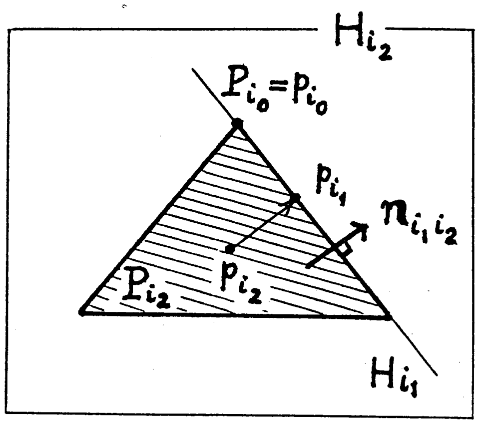
$$[i] = \begin{cases} 1 & (k = 0), \\ [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k] & (k = 1, 2, \ldots, n). \end{cases}$$

Let $h_m^{(j)}(\xi)$ be the complete symmetric polynomial of degree m in j-variables:

$$h_m^{(j)}(\xi_1, \ldots, \xi_j) = \sum_{m_1 + \cdots + m_j = m} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_j^{m_j},$$

where the summation is taken over all j-tuples $(m_1, \ldots, m_j)$ of nonnegative integers satisfying the indicated condition. Finally, we set $\langle \xi, \eta \rangle = \xi_1 \eta_1 + \cdots + \xi_n \eta_n$ for two complex vectors $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$.





The following theorem gives a characterization of the $P(k)$-mean value property in terms of a system of partial differential equations.

Theorem 2.1. *Any $f \in \mathcal{H}_{P(k)}(\Omega)$ is smooth in Ω and satisfies the system of partial differential equations:*

$$(*) \qquad \tau_m^{(k)}(\partial) f = 0 \qquad (m = 1, 2, 3, \ldots),$$

where $\tau_m^{(k)}(\xi)$ is the homogeneous polynomial of degree m defined by

$$\tau_m^{(k)}(\xi) = \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \ldots, \langle p_{i_k}, \xi \rangle),$$

Conversely, any weak solution of $(*)$ is real analytic and belongs to $\mathcal{H}_{P(k)}(\Omega)$.

The system $(*)$ enjoys the following remarkable property.

Theorem 2.2. *The system $(*)$ is holonomic.*

The holonomicity follows from the geometry and combinatorics of the polytope P. Theorems 2.1 and 2.2 play an essential role in establishing Theorem 1.3.

3. Polytopes with Symmetry

Our problem is of particular interest if P admits symmetry. Let $G \subset O(n)$ be the complete symmetry group of P. Then the following theorem gives a lower bound of the dimension of $\mathcal{H}_{P(k)}(\Omega)$ in terms of G.

Theorem 3.1. $\dim \mathcal{H}_{P(k)}(\Omega) \geq |G|$.

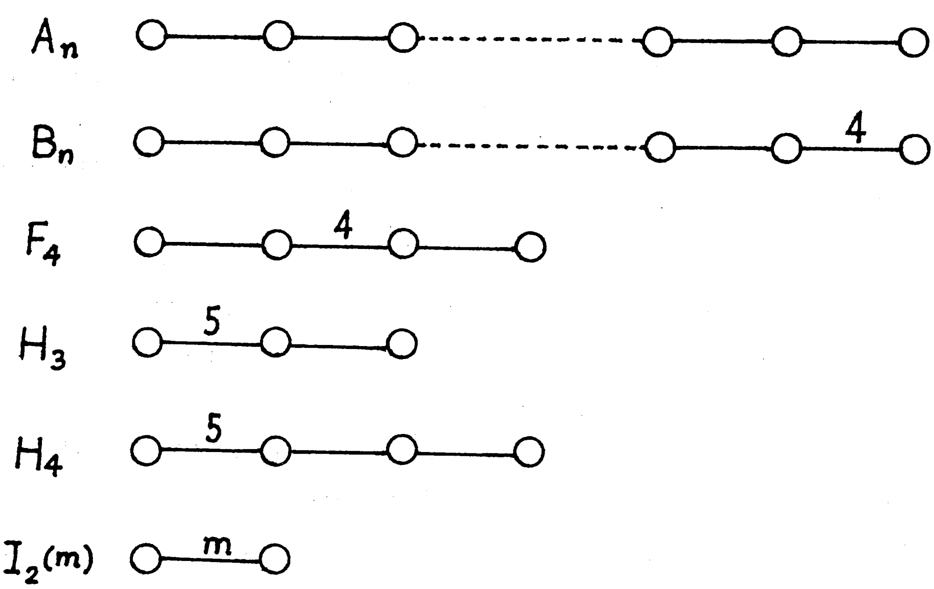
The relations between the $P(k)$-harmonic functions and the symmetry of P must be investigated more thoroughly.

We turn our attention to more specific polytopes. For any regular convex polytope P, we are able to determine the function space $\mathcal{H}_{P(k)}$ explicitly. We begin with the classification of regular convex polytopes. The complete symmetry group $G \subset O(n)$ of P is an irreducible finite reflection group. All irreducible finite reflection groups are classified in terms of connected Coxeter graphs (see e.g. [9]). Thus we have the following diagram.

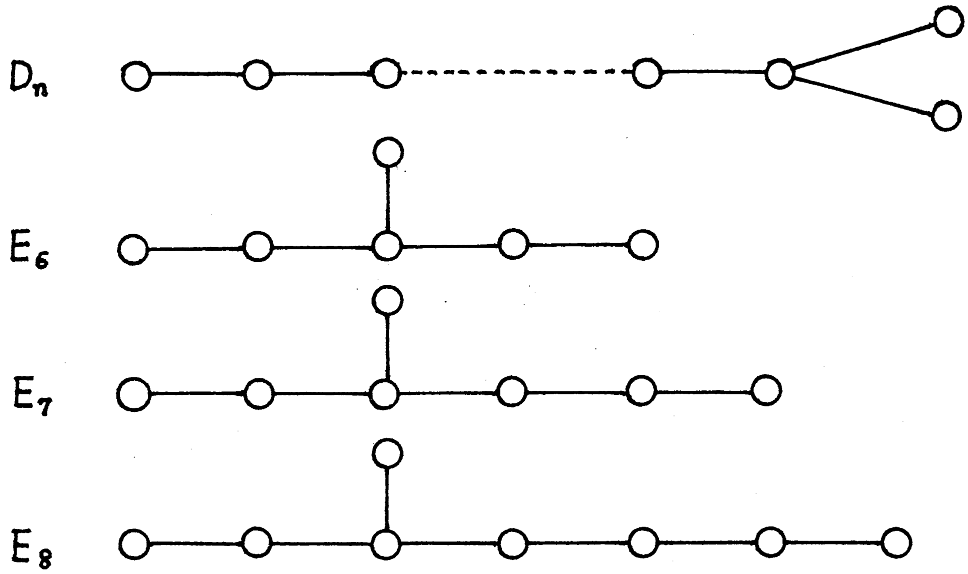
Diagram 3.2.

$$\begin{array}{ccccl}
\{\text{regular convex polytopes}\} & \ni & P & & \\
\downarrow & & \downarrow & & \text{(symmetry group)} \\
\{\text{irreducible finite reflection groups}\} & \ni & G & & \\
\updownarrow & & \updownarrow & & \text{(one-to-one)} \\
\{\text{connected Coxeter graphs}\} & \ni & \Gamma & &
\end{array}$$

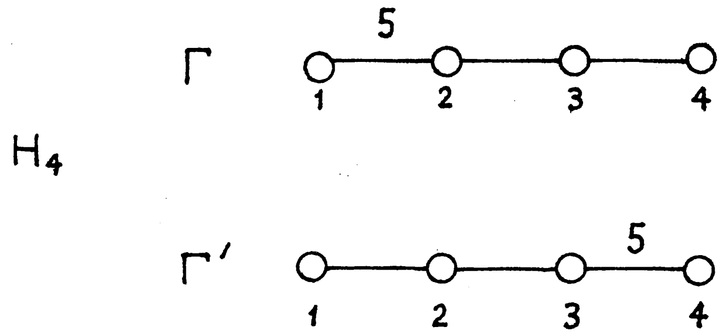
An irreducible finite reflection group G is the complete symmetry group of a regular convex polytope P if and only if the Coxeter graph Γ of G has no node. Therefore all admissible graphs are precisely those of types A_n, B_n, F_4, H_3, H_4 and $I_2(m)$.



Graphs of types D_n, E_6, E_7 and E_8 do not correspond to any regular convex polytope.



A regular convex polytope P and its dual P^* correspond to the same Coxeter graph Γ, but no other regular convex polytopes correspond to Γ. Moreover, P is *self-dual* if and only if $\Gamma = \Gamma'$, where Γ' is the *reversed* graph of Γ.



Accordingly, we have the following classification of regular convex polytopes.

Classification of regular convex polytopes. ($n = \dim P$)

(1) A_n (regular simplexes[1], self-dual),
(2) B_n (cross polytopes and measure polytopes[2]),
(3) H_3 (icosahedron and dodecahedron),
(4) H_4 (600-cells and 120-cells),
(5) F_4 (24-cells, self-dual), and
(6) $I_2(m)$ (regular m-gon, self-dual).

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[1]tetrahedron for $n = 3$

[2]octahedron and cube for $n = 3$

Theorem 3.3. *Let P be any regular convex polytope in $\mathbb{R}^n$ with center at the origin. Let $G \subset O(n)$ be the complete symmetry group of P. Then, for any $k = 0, 1, \ldots, n$,*

$$\dim \mathcal{H}_{P(k)} = |G|,$$
$$\mathcal{H}_{P(k)} = \mathbb{C}[\partial]\Delta(x),$$

where $\Delta(x)$ is the fundamental alternating polynomial of the reflection group G.

See also [1][2][4][6][7][14], where a part of Theorem 3.3 has already been obtained. But our treatment is completely different and much more thorough, and the result is a final one.

Theorem 3.3 reminds us of the G-harmonic functions due to Steinberg [14]. For a finite subgroup G of $GL(n, \mathbb{R})$, let R be the ring of G-invariant polynomials, R_+ the maximal ideal of R consistsing of all elements $\phi \in R$ such that $\phi(0) = 0$. Then $f \in C^\infty(\mathbb{R}^n)$ is said to be *G-harmonic* if f satisfies the system of partial differential equations:

$$\phi(\partial)f = 0 \qquad (\phi \in R_+).$$

Let $\mathcal{H}_G$ denote the set of all G-harmonic functions. It is known that $\mathcal{H}_G$ is a finite-dimensional linear space of polynomials (see [14]). Now Theorem 3.3 is restated as follows.

Theorem 3.4. *Let P be any regular convex polytope in $\mathbb{R}^n$ with center at the origin, and $G \subset O(n)$ be its complete symmetry group. Then,*

$$\mathcal{H}_{P(k)} = \mathcal{H}_G \qquad (k = 0, 1, \ldots, n).$$

Invariant theory for finite reflection groups, as well as systems of invariant differential equations plays an essential role in establishing Theorem 3.4. In the course of the proof, we were able to introduce a distinguished basis of G-invarinat polynomials (canonically attached to the invarinat differential equations) for each finite reflection group G (see [11]).

4. Open Problem

There is an open problem which has constantly interested the author. A polytope P is said to admits *ample symmetry* if the complete symmetry group G of P is irreducible. Recall that if P admits ample symmetry, then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of *harmonic polynomials* (see (5) of Theorem 1.3). So the following problem naturally occurs to us.

Problem 4.1. Is there any infinite sequence $P_1, P_2, \ldots, P_m, \ldots$ of polytopes in $\mathbb{R}^n$ with ample symmetry such that, for any/some $k = 0, 1, \ldots, n$, the following properties hold:

(1) the polytopes P_m approximate the unit ball B^n as $m \to \infty$,
(2) $\mathcal{H}_{P_1(k)} \subset \mathcal{H}_{P_2(k)} \subset \cdots \subset \mathcal{H}_{P_m(k)} \subset \cdots$,
(3) the spaces $\mathcal{H}_{P_m(k)}$ exhaust the set of all harmonic polynomials in n-variables as $m \to \infty$.

In the case of two-dimension, we know that the answer is *yes*. Indeed, in view of Example 1.5, we can take P_m to be a regular convex m-gon $(m = 3, 4, 5, \dots)$. However, the problem becomes quite difficult if the dimension n is greater than two. At present the author has no substantial idea to tackle it. The difficulty lies in the fact that if n is greater than two, then there are only finitely many irreducible finite subgroups of $O(n)$ up to conjugacy. Therefore group theoretical approach based on the symmetry of polytopes is not sufficient for solving the problem. We hope that a completely new idea is introduced in the future.

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